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Continuous, Discrete, Ultradiscrete Waves

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1 Introduction to Ultradiscretization

'Ultradiscretization' is a technique to discretize dependent variable of difference equations. Using this technique, we can obtain universal mathematical structure among differential equations, difference equations and digital equations. The following is one example on the diffusion equation,

$$g_t = g_{xx}. \tag{1}$$

We have a difference analogue to the above equation,

$$f_j^{n+1} = \frac{1}{2}(f_{j+1}^n + f_{j-1}^n). \tag{2}$$

If we use the following transformation,

$$f_j^n = \exp(F_j^n/\varepsilon), \tag{3}$$

we obtain from (2),

$$F_j^{n+1} = \varepsilon \log(e^{F_{j+1}^n/\varepsilon} + e^{F_{j-1}^n/\varepsilon}) - \varepsilon \log 2. \tag{4}$$

Taking $\varepsilon \to +0$ and using an identity

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) = \max(A, B, \cdots), \tag{5}$$

we obtain

$$F_j^{n+1} = \max(F_{j+1}^n, F_{j-1}^n). \tag{6}$$

Note that F_j^n are always integer if initial F are all integer. In this meaning, we discretize a dependent variable in (2) and obtain (6).

In the above discretizing process, we only use a transformation of variable (3) and an identity (5). Therefore, this technique can be applied to other equations including nonlinear ones. We call this kind of discretizing process on dependent variable 'ultradiscretization'.

The second example is Burgers equation. It is well known Burgers equation can be linearized through Cole–Hopf transformation.

$$\begin{cases} g_t = g_{xx} \qquad \text{(diffusion eq.)} \\ \qquad \updownarrow \quad v = (\log g)_x \qquad \text{(Cole–Hopf trans.)} \\ v_t = 2vv_x + v_{xx} \qquad \text{(Burgers eq.)} \end{cases} \tag{7}$$

We can obtain a difference analogue to the above system,

$$\begin{cases} f_j^{n+1} = \dfrac{1}{2}(f_{j+1}^n + f_{j-1}^n) \qquad \text{(difference diffusion eq.)} \\ \qquad \updownarrow \quad u_j^n = \dfrac{1}{\Delta x}(\log f_{j+1}^n - \log f_j^n) \qquad \text{(difference Cole–Hopf trans.)} \\ u_j^{n+1} = u_j^n + \dfrac{1}{\Delta x}\{\log(e^{-\Delta x u_j^n} + e^{\Delta x u_{j+1}^n}) \\ \qquad\qquad - \log(e^{-\Delta x u_{j-1}^n} + e^{\Delta x u_j^n})\} \qquad \text{(difference Burgers eq.)} \end{cases} \tag{8}$$

Moreover, if we use the following transformations of variables,

$$f_j^n = \exp(F_j^n/\varepsilon), \qquad \Delta x\, u_j^n = (U_j^n - \frac{1}{2})/\varepsilon, \tag{9}$$

and take a limit $\varepsilon \to +0$, we obtain

$$\begin{cases} F_j^{n+1} = \max(F_{j+1}^n, F_{j-1}^n) \qquad \text{(ultradiscrete diffusion eq.)} \\ \qquad \updownarrow \quad U_j^n = F_{j+1}^n - F_j^n + 1/2 \qquad \text{(ultradiscrete Cole–Hopf trans.)} \\ U_j^{n+1} = U_j^n + \min(U_{j-1}^n, 1 - U_j^n) - \min(U_j^n, 1 - U_{j+1}^n) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{(ultradiscrete Burgers eq.)} \end{cases} \tag{10}$$

If all initial U's are integer in u-Burgers eq, U's at any time become integer. (In this case, F may be half integer but is also discretized.)

Moreover, assuming that initial U's are all 0 or 1, we can easily show U's at any time also become 0 or 1. Under this restriction of values, we can consider that u-Burgers equation is a cellular automaton (CA) with state values 0 and 1. This CA follows a time evolution rule,

$$\frac{U_{j-1}^n U_j^n U_{j+1}^n}{U_j^{n+1}} : \frac{000}{0} \; \frac{001}{0} \; \frac{010}{0} \; \frac{011}{1} \; \frac{100}{1} \; \frac{101}{1} \; \frac{110}{0} \; \frac{111}{1}, \tag{11}$$

and it is equivalent to the rule–184 elementary CA after Wolfram. Note that there is a common linearization structure among Burgers, d-Burgers and u-Burgers equations, we can propose their explicit solutions through diffusion equation, even to u-Burgers equation. Especially, we can easily show a pattern selection mechanism of rule–184 CA using the linearization.

We have already obtained various ultradiscretizable nonlinear equations and interesting results. Ultradiscretization gives a universal view on systems from continuous to digital, and proposes a new approach to analyze their mathematical structure.

2 Max–Plus Equation on Pattern Formation

In the previous section, we introduced successful examples of ultradiscretization method. However, there is a weak point in the method. We usually use transformation of variables like (3) in the ultradiscretization. Then, 'positivity' of dependent variable is necessary for difference equations and not all of difference equations can be automatically ultradiscretized. When we meet this difficulty, one of possible solutions is 'back-ultradiscretization', that is, to make a CA or ultradiscrete equation first, follow the reverse path of ultradiscretization, and obtain a continuous (differential or difference) equation.

There occurs another difficulty in this approach though back-ultradiscretization is always easy. Because the continuous equation obtained often becomes trivial. However, when there is no clue to digitalize a system, this approach is always valuable to try.

In this section, we show a digital equation relevant to pattern formation system. This equation is expressed by max and addition operators. Since ultradiscrete equations are always expressed by max and addition, they are called 'max–plus equation' after max–plus algebra. Since we have not yet obtained a successful back-ultradiscretization of the digital equation shown below, we call it max–plus equation, not ultradiscrete equation.

The equation is

$$u_{ij}^{t+1} = \max(u_{i+1j}^t, u_{i-1j}^t, u_{ij+1}^t, u_{ij-1}^t, u_{ij}^t) - u_{ij}^t - u_{ij}^{t-1}, \tag{12}$$

where i and j are space lattices and t is integer time. This equation is second order in time and we can rewrite the above equation to a system of equations of first order using $v_{ij}^t = u_{ij}^{t-1}$,

$$\begin{cases} u_{ij}^{t+1} = \max(u_{i+1j}^t, u_{i-1j}^t, u_{ij+1}^t, u_{ij-1}^t, u_{ij}^t) - u_{ij}^t - v_{ij}^t \\ v_{ij}^{t+1} = u_{ij}^t \end{cases} \tag{13}$$

In this form, we may consider that it is an activator–inhibitor system. The reason is as follows:(i) u has a diffusion effect by max term. (ii) u decreases by v. (iii) v is equal to u at a previous time step, that is, v increases if u increases. (iv) Thus u is activator and v is inhibitor.

Figure 1 shows two numerical results, Fig. 1 (a) shows a target pattern and (b) a spiral pattern. This pattern is often observed in some of typical pattern formation systems, for example, reaction-diffusion systems. However, there are two remarkable features different from usual reaction-diffusion systems.

One is that (12) is reversible in time and the other is that if u is a solution, $C\,u$ is also. Therefore, we consider that a usual reaction-diffusion system can not be obtained by back-ultradiscretization of (12). Thus there occurs a question 'What is a continuous system corresponding to (12)?'

We have not yet succeeded non-trivial back-ultradiscretization of (12), we can not answer this question now. However, the above system has a remarkable feature. Let us prepare a family of curves shown in Fig. 2 (a) and (b) corresponding to Fig. 1 (a) and (b). Each curve is labeled with an integer as shown in figures. Then, assume values of u's on curve n are the same and f_n^t denotes the value on curve n at time t. Then, from a symmetry of (12), (12) reduces to the following one-dimensional equation for both patterns:

$$f_n^{t+1} = \max(f_{n+1}^t, f_n^t, f_{n-1}^t) - f_n^t - f_n^{t-1}. \qquad (14)$$

Note that

$$f_0^{t+1} = \max(f_0^t, f_1^t) - f_0^t - f_0^{t-1} \qquad (15)$$

is applied for $n = 0$ as a boundary condition in the target pattern and n becomes a periodic lattice with period 4 in the spiral pattern. Under this difference on boundary condition, both patterns satisfy the same one-dimensional equation exactly.

Moreover, both patterns become a traveling wave solution in (14). Therefore, we can make a reduction further. If we set a traveling wave form $f_n^t = g_{n-t}$, we obtain

$$g_{n+1} = \max(g_{n+1}, g_n, g_{n-1}) - g_n - g_{n-1} \qquad (16)$$

from (14). We would emphasize the followings:(i) Such exact reduction is very difficult in continuous reaction-diffusion systems. (ii) The families of curves in Fig. 2 can be considered to be digital analogues to circles and spirals in a continuous coordinate system. These points suggest (12) is related to a solvable continuous pattern model. It is an interesting future problem to find a continuous correspondence.

```
1 . . 1 . . 1 . 1 . . 1 . . 1
. . 1 . . 1 . . . 1 . . 1 . .
. 1 . . 1 . . 1 . . 1 . . 1 .
1 . . 1 . . 1 . 1 . . 1 . . 1
. . 1 . . 1 . . . 1 . . 1 . .
. 1 . . 1 . . 1 . . 1 . . 1 .
1 . . 1 . . 1 . 1 . . 1 . . 1
. . 1 . . 1 . 1 . 1 . . 1 . .
1 . . 1 . . 1 . 1 . . 1 . . 1
. 1 . . 1 . . 1 . . 1 . . 1 .
. . 1 . . 1 . . . 1 . . 1 . .
1 . . 1 . . 1 . 1 . . 1 . . 1
. 1 . . 1 . . 1 . . 1 . . 1 .
. . 1 . . 1 . . . 1 . . 1 . .
1 . . 1 . . 1 . 1 . . 1 . . 1
```

t

```
. . 1 . . 1 . . . 1 . . 1 . .
. 1 . . 1 . . 1 . . 1 . . 1 .
1 . . 1 . . 1 . 1 . . 1 . . 1
. . 1 . . 1 . . . 1 . . 1 . .
. 1 . . 1 . . 1 . . 1 . . 1 .
1 . . 1 . . 1 . 1 . . 1 . . 1
. . 1 . . 1 . . . 1 . . 1 . .
. 1 . . 1 . . 1 . . 1 . . 1 .
. . 1 . . 1 . . . 1 . . 1 . .
1 . . 1 . . 1 . 1 . . 1 . . 1
. 1 . . 1 . . 1 . . 1 . . 1 .
. . 1 . . 1 . . . 1 . . 1 . .
1 . . 1 . . 1 . 1 . . 1 . . 1
. 1 . . 1 . . 1 . . 1 . . 1 .
. . 1 . . 1 . . . 1 . . 1 . .
```

$t+1$

```
. 1 . . 1 . . 1 . . 1 . . 1 .
1 . . 1 . . 1 . 1 . . 1 . . 1
. . 1 . . 1 . . . 1 . . 1 . .
. 1 . . 1 . . 1 . . 1 . . 1 .
1 . . 1 . . 1 . 1 . . 1 . . 1
. . 1 . . 1 . . . 1 . . 1 . .
. 1 . . 1 . . 1 . . 1 . . 1 .
1 . . 1 . . 1 a 1 . . 1 . . 1
. 1 . . 1 . . 1 . . 1 . . 1 .
. . 1 . . 1 . . . 1 . . 1 . .
1 . . 1 . . 1 . 1 . . 1 . . 1
. 1 . . 1 . . 1 . . 1 . . 1 .
. . 1 . . 1 . . . 1 . . 1 . .
1 . . 1 . . 1 . 1 . . 1 . . 1
. 1 . . 1 . . 1 . . 1 . . 1 .
```

$t+2$

Figure 1 (a) : Target pattern. '.' and 'a' denote 0 and -1 respectively. This solution is periodic with a period of 3 time steps.

```
. 1 . . . 1 . . . . . 1 . . .
1 . . . 1 . . . 1 . . . 1 . .
. . . 1 . . . 1 . 1 . . . 1 .
. . 1 . . . 1 . . . 1 . . . 1
. 1 . . . 1 . . . . . 1 . . .
1 . . . 1 . . . 1 . . . 1 . .
. . . 1 . . . 1 . 1 . . . 1 .
. . 1 . . . 1 . . . 1 . . . 1
. . . 1 . . . 1 . . . 1 . . .
1 . . . 1 . . . . . 1 . . . 1
. 1 . . . 1 . . . 1 . . . 1 .
. . 1 . . . 1 . 1 . . . 1 . .
. . . 1 . . . 1 . . . 1 . . .
1 . . . 1 . . . . . 1 . . . 1
. 1 . . . 1 . . . 1 . . . 1 .
```

t

```
1 . . . 1 . . . 1 . . . 1 . .
. . . 1 . . . 1 . 1 . . . 1 .
. . 1 . . . 1 . . . 1 . . . 1
. 1 . . . 1 . . . . . 1 . . .
1 . . . 1 . . . 1 . . . 1 . .
. . . 1 . . . 1 . 1 . . . 1 .
. . 1 . . . 1 . . . 1 . . . 1
. 1 . . . 1 . . . . . 1 . . .
. . 1 . . . 1 . 1 . . . 1 . .
. . . 1 . . . 1 . . . 1 . . .
1 . . . 1 . . . . . 1 . . . 1
. 1 . . . 1 . . . 1 . . . 1 .
. . 1 . . . 1 . 1 . . . 1 . .
. . . 1 . . . 1 . . . 1 . . .
1 . . . 1 . . . . . 1 . . . 1
```

$t+1$

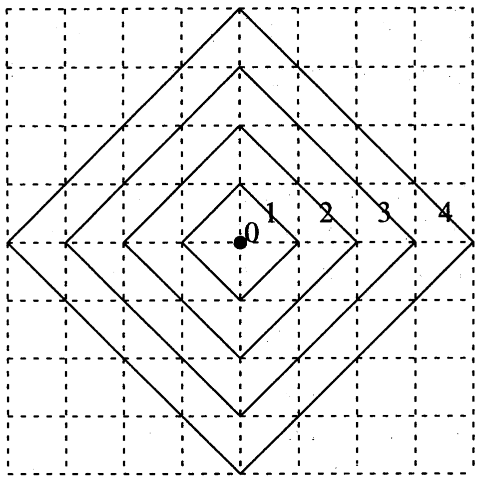
```
. . . 1 . . . 1 . 1 . . . 1 .
. . 1 . . . 1 . . . 1 . . . 1
. 1 . . . 1 . . . . . 1 . . .
1 . . . 1 . . . 1 . . . 1 . .
. . . 1 . . . 1 . 1 . . . 1 .
. . 1 . . . 1 . . . 1 . . . 1
. 1 . . . 1 . . . . . 1 . . .
1 . . . 1 . . . 1 . . . 1 . .
. 1 . . . 1 . . . 1 . . . 1 .
. . 1 . . . 1 . 1 . . . 1 . .
. . . 1 . . . 1 . . . 1 . . .
1 . . . 1 . . . . . 1 . . . 1
. 1 . . . 1 . . . 1 . . . 1 .
. . 1 . . . 1 . 1 . . . 1 . .
. . . 1 . . . 1 . . . 1 . . .
```

$t+2$

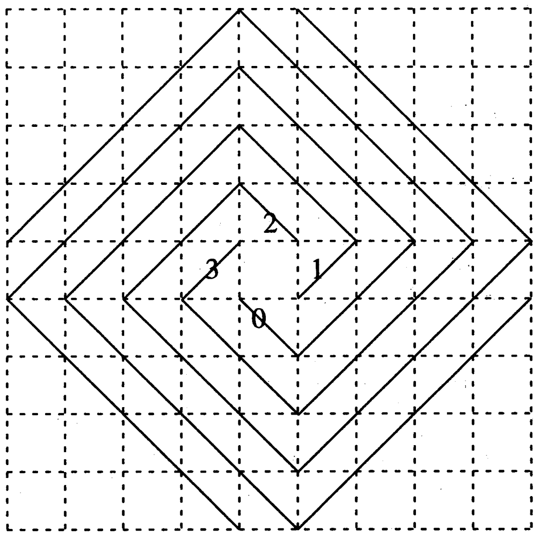
```
. . 1 . . . 1 . . . 1 . . . 1
. 1 . . . 1 . . . . . 1 . . .
1 . . . 1 . . . 1 . . . 1 . .
. . . 1 . . . 1 . 1 . . . 1 .
. . 1 . . . 1 . . . 1 . . . 1
. 1 . . . 1 . . . . . 1 . . .
1 . . . 1 . . . 1 . . . 1 . .
. . . 1 . . . 1 . 1 . . . 1 .
1 . . . 1 . . . . . 1 . . . 1
. 1 . . . 1 . . . 1 . . . 1 .
. . 1 . . . 1 . 1 . . . 1 . .
. . . 1 . . . 1 . . . 1 . . .
1 . . . 1 . . . . . 1 . . . 1
. 1 . . . 1 . . . 1 . . . 1 .
. . 1 . . . 1 . 1 . . . 1 . .
```

$t+3$

Figure 1 (b) : Spiral pattern. '.' denotes 0. This solution is periodic with a period of 4 time steps.



(a)



(b)

Figure 2: Family of curves. (a) and (b) correspond to patterns in Fig. 1 (a) and (b) respectively.